

On finite reflexive homomorphism-homogeneous binary relational systems[☆]

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Abstract

A structure is called homogeneous if every isomorphism between finitely induced substructures of the structure extends to an automorphism of the structure. Recently, P. J. Cameron and J. Nešetřil introduced a relaxed version of homogeneity: we say that a structure is homomorphism-homogeneous if every homomorphism between finitely induced substructures of the structure extends to an endomorphism of the structure.

In this paper we consider finite homomorphism-homogeneous relational systems with one reflexive binary relation. We show that for a large part of such relational systems (bidirectionally connected digraphs; a digraph is bidirectionally connected if each of its connected components can be traversed by \rightleftarrows -paths) the problem of deciding whether the system is homomorphism-homogeneous is coNP-complete. Consequently, for this class of relational systems we cannot hope for a description involving a catalogue, where by a catalogue we understand a finite list of polynomially decidable classes of structures. On the other hand, in case of bidirectionally disconnected digraphs we present the full characterization. Our main result states that if a digraph is bidirectionally disconnected, then it is homomorphism-homogeneous if and only if it is either a finite homomorphism-homogeneous quasiorder, or an inflation of a homomorphism-homogeneous digraph with involution (a peculiar class of digraphs introduced later in the paper), or an inflation of a digraph whose only connected components are C_3° and $\mathbf{1}^\circ$.

[☆]Supported by the Grant No 144017 of the Ministry of Science of the republic of Serbia
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1. Introduction

A structure is *homogeneous* if every isomorphism between finitely induced substructures of the structure extends to an automorphism of the structure. In their recent paper [1] the authors discuss a generalization of homogeneity to various types of morphisms between structures, and in particular introduce the notion of homomorphism-homogeneous structures:

Definition 1.1 (Cameron, Nešetřil [1]) A structure is called *homomorphism-homogeneous* if every homomorphism between finitely induced substructures of the structure extends to an endomorphism of the structure.

This paper grew out of the authors' intention to characterize all finite homomorphism-homogeneous relational systems with one reflexive binary relation (binary relational systems). However, the complete characterization of such relational systems turns out to be rather involved since the presence of loops allows homomorphisms to spread their wings. What makes the problem unsolvable in general is a result presented in [5] where the authors show that the problem of deciding whether a finite graph with loops allowed is homomorphism-homogeneous is coNP-complete.

After the introductory Section 2, in Section 3 we adapt the argument of [5] to show that the same holds even for bidirectionally connected improper digraphs (a digraph is bidirectionally connected if each of its connected components can be traversed by \rightleftarrows -paths; it is improper if it contains both edges of the form \rightleftarrows and of the form \rightarrow). The fact that deciding homomorphism-homogeneity is computationally hard for bidirectionally connected digraphs means that for this class of digraphs we cannot hope for a full description that involves a catalogue, where by a catalogue we understand a finite list of polynomially decidable classes of structures. We then turn to the classification of bidirectionally disconnected systems, which heavily relies on a peculiar class of digraphs we refer to as digraphs with involution. Section 4 is devoted to the classification of homomorphism-homogeneous digraphs in that class. A rather long Section 5 concludes the paper and classifies all finite reflexive homomorphism-homogeneous bidirectionally disconnected systems. Our main result is Corollary 5.12 which states that if a digraph is bidirectionally disconnected, then it is homomorphism-homogeneous if and only if

it is either a finite homomorphism-homogeneous quasiorder, or an inflation of a homomorphism-homogeneous digraph with involution, or an inflation of a digraph whose only connected components are C_3° and $\mathbf{1}^\circ$.

2. Preliminaries

A binary relational system is an ordered pair (V, E) where $E \subseteq V^2$ is a binary relation on V . A binary relational system (V, E) is *reflexive* if $(x, x) \in E$ for all $x \in V$, *irreflexive* if $(x, x) \notin E$ for all $x \in V$, *symmetric* if $(x, y) \in E$ implies $(y, x) \in E$ for all $x, y \in V$ and *antisymmetric* if $(x, y) \in E$ implies $(y, x) \notin E$ for all distinct $x, y \in V$.

Binary relational systems can be thought of in terms of digraphs (hence the notation (V, E)). Then V is the set of *vertices* and E is the set of *edges* of the binary relational system/digraph (V, E) . Edges of the form (x, x) are called *loops*. If $(x, x) \in E$ we also say that x *has a loop*. Instead of $(x, y) \in E$ we often write $x \rightarrow y$ and say that x *dominates* y , or that y *is dominated by* x . By $x \sim y$ we denote that $x \rightarrow y$ or $y \rightarrow x$, while $x \rightrightarrows y$ denotes that $x \rightarrow y$ and $y \rightarrow x$. If $x \rightrightarrows y$, we say that x and y form a *double edge*. We shall also say that a vertex x is *incident with a double edge* if there is a vertex $y \neq x$ such that $x \rightrightarrows y$.

Digraphs (V, E) where E is a symmetric binary relation on V are usually referred to as *graphs*. *Proper digraphs* are digraphs (V, E) where E is an antisymmetric binary relation. In this paper, digraphs (V, E) where E is neither antisymmetric nor symmetric will be referred to as *improper digraphs*. In an improper digraph there exists a pair of distinct vertices x and y such that $x \rightrightarrows y$ and another pair of distinct vertices u and v such that $u \rightarrow v$ and $v \not\rightarrow u$.

If $X, Y \subseteq V$ are nonempty subsets of V then $X \rightarrow Y$ means that $x \rightarrow y$ for some $x \in X$ and some $y \in Y$. By $X \sim Y$ we denote that $X \rightarrow Y$ or $Y \rightarrow X$, while $X \rightrightarrows Y$ denotes that $X \rightarrow Y$ and $Y \rightarrow X$. Moreover, $X \Rightarrow Y$ stands for $x \rightarrow y$ for all $x \in X$ and $y \in Y$. Instead of $\{x\} \Rightarrow Y$ and $X \Rightarrow \{y\}$ we write $x \Rightarrow Y$ and $X \Rightarrow y$, respectively, and analogously for $x \rightarrow Y$, $X \rightarrow y$ and $x \rightrightarrows Y$. Let $r, s, t, u \in V(D)$ be vertices of a digraph D . We write $\{r, s\} \bowtie \{t, u\}$ to denote that $r \rightarrow t \rightarrow s \rightarrow u \rightarrow r$ or $r \rightarrow u \rightarrow s \rightarrow t \rightarrow r$.

A digraph $D' = (V', E')$ is a *subdigraph* of a digraph $D = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We write $D' \leq D$ to denote that D' is isomorphic

to a subdigraph of D . For $\emptyset \neq W \subseteq V$ by $D[W]$ we denote the digraph $(W, E \cap W^2)$ which we refer to as the *subdigraph of D induced by W* .

Vertices x and y are *connected in D* if there exists a sequence of vertices $z_1, \dots, z_k \in V$ such that $x = z_1 \sim \dots \sim z_k = y$. A digraph D is *weakly connected* if each pair of distinct vertices of D is connected in D . A digraph D is *disconnected* if it is not weakly connected. A *connected component of D* is a maximal set $S \subseteq V$ such that $D[S]$ is weakly connected. The number of connected components of D will be denoted by $\omega(D)$.

Vertices x and y are *bidirectionally connected in D* if there exists a sequence of vertices $z_1, \dots, z_k \in V$ such that $x = z_1 \rightleftarrows \dots \rightleftarrows z_k = y$. Define a binary relation $\theta(D)$ on $V(D)$ as follows: $(x, y) \in \theta(D)$ if and only if $x = y$ or x and y are bidirectionally connected. Clearly, $\theta(D)$ is an equivalence relation on $V(D)$ and $\omega(D) \leq |V(D)/\theta(D)|$. We say that a digraph D is *bidirectionally connected* if $\omega(D) = |V(D)/\theta(D)|$, and that it is *bidirectionally disconnected* if $\omega(D) < |V(D)/\theta(D)|$. Note that a bidirectionally connected digraph need not be connected, and that a bidirectionally disconnected digraph need not be disconnected; a digraph D is bidirectionally connected if every connected component of D contains precisely one $\theta(D)$ -class, while it is bidirectionally disconnected if there exists a connected component of D which consists of at least two $\theta(D)$ -classes. In particular, every proper digraph with at least two vertices is bidirectionally disconnected, and every graph (even a disconnected one) is bidirectionally connected.

Let K_n denote the complete irreflexive graph on n vertices, and let K_n° denote the complete reflexive graph on n vertices. Let $\mathbf{1}$ denote the trivial digraph with only one vertex and no edges, and let $\mathbf{1}^\circ$ denote the digraph with only one vertex with a loop. An *oriented cycle with n vertices* is a digraph C_n whose vertices are $1, 2, \dots, n$, $n \geq 3$, and whose only edges are $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$.

For digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$, by $D_1 + D_2$ we denote the *disjoint union* of D_1 and D_2 . We assume that $D + O = O + D = D$, where $O = (\emptyset, \emptyset)$ denotes the *empty digraph*. The disjoint union $\underbrace{D + \dots + D}_k$

consisting of $k \geq 1$ copies of D will be abbreviated to $k \cdot D$. Moreover, we let $0 \cdot D = O$.

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. We say that $f : V_1 \rightarrow V_2$ is a *homomorphism* between D_1 and D_2 and write $f : D_1 \rightarrow D_2$ if

$$x \rightarrow y \text{ implies } f(x) \rightarrow f(y), \text{ for all } x, y \in V_1.$$

An *endomorphism* is a homomorphism from D into itself. A mapping $f : V_1 \rightarrow V_2$ is an *isomorphism* between D_1 and D_2 if f is bijective and

$$x \rightarrow y \text{ if and only if } f(x) \rightarrow f(y), \text{ for all } x, y \in V_1.$$

Digraphs D_1 and D_2 are *isomorphic* if there is an isomorphism between them. We write $D_1 \cong D_2$. An *automorphism* is an isomorphism from D onto itself.

A digraph D is *homomorphism-homogeneous* if every homomorphism $f : W_1 \rightarrow W_2$ between finitely induced subdigraphs of D extends to an endomorphism of D (see Definition 1.1).

For digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ we write $D_1 \Rightarrow D_2$ to denote that every homomorphism $f : D_1[U] \rightarrow D_2[W]$, where $\emptyset \neq U \subseteq V_1$ and $\emptyset \neq W \subseteq V_2$, extends to a homomorphism $f^* : D_1 \rightarrow D_2$. Clearly, a digraph D is homomorphism-homogeneous if and only if $D \Rightarrow D$. The following two statements are obvious:

Lemma 2.1 *Let D be a digraph. Then D is homomorphism-homogeneous if and only if $D[S] \Rightarrow D[S']$ for every pair of (not necessarily distinct) connected components S, S' of D .*

Lemma 2.2 *Let D be an improper digraph and let f be an endomorphism of D . Then*

- *for every connected component S of D there exists a connected component S' of D such that $f(S) \subseteq S'$;*
- *for every $S \in V(D)/\theta(D)$ there exists an $S' \in V(D)/\theta(D)$ such that $f(S) \subseteq S'$.*

A digraph $D = (V, E)$ is *transitive* if $x \rightarrow y \rightarrow z$ implies $x \rightarrow z$ for all $x, y, z \in V$. Transitive reflexive proper digraphs are usually referred to as *partially ordered sets*. Recall that a mapping $f : A \rightarrow B$ is a *homomorphism* between partially ordered sets (A, \leq) and (B, \leq) if

$$x \leq y \text{ implies } f(x) \leq f(y), \text{ for all } x, y \in A.$$

It is clear that a mapping is a homomorphism between two partially ordered sets in the above sense if and only if the mapping is a homomorphism between the corresponding digraphs. Therefore, a partially ordered set is homomorphism-homogeneous as a partially ordered set if and only if it is homomorphism-homogeneous as a digraph.

Theorem 2.3 ([3]) *A finite partially ordered set (A, \leq) is homomorphism-homogeneous if and only if*

- (1) *every connected component of (A, \leq) is a chain (this case includes anti-chains);*
- (2) *(A, \leq) is a tree, where a tree is a connected partially ordered set whose every up-set $\uparrow x$ is a chain;*
- (3) *(A, \leq) is a dual tree, where a dual tree is a connected partially ordered set whose every down-set $\downarrow x$ is a chain;*
- (4) *(A, \leq) splits into a tree and a dual tree in the following sense: there exists a partition $\{I, F\}$ of A such that*
 - (i) *I is an ideal in A and a tree,*
 - (ii) *F is a filter in A and a dual tree, and*
 - (iii) *$\forall x \in I \exists y \in F (x \leq y)$ and $\forall y \in F \exists x \in I (x \leq y)$;*
- (5) *(A, \leq) is a lattice.*

Reflexive finite homomorphism-homogeneous proper digraphs were characterized in Theorem 3.10 of [4]:

Theorem 2.4 ([4]) *Let D be a reflexive proper digraph. Then D is homomorphism-homogeneous if and only if D is one of the following digraphs:*

- (1) *$k \cdot C_3^\circ + l \cdot \mathbf{1}^\circ$ for some $k, l \geq 0$ such that $k + l \geq 1$;*
- (2) *a finite homomorphism-homogeneous partially ordered set (see Theorem 2.3).*

3. Bidirectionally connected systems

Rusinov and Schweitzer have shown in [5] that the problem of deciding whether a finite graph with loops allowed is homomorphism-homogeneous is coNP-complete. In this section we adapt the argument of [5] to show that the same holds for bidirectionally connected improper digraphs. Consequently, for the class of bidirectionally connected digraphs we cannot hope for a full description that involves a catalogue.

Let $M = \{\leftarrow, \rightarrow, \rightleftarrows\}$ and let $\mathcal{M} = (M, \leq)$ be the three-element partially ordered set depicted in Fig. 1. Let D be an improper digraph. We say that

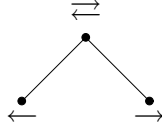


Figure 1: The poset \mathcal{M}

a vertex $c \in V(D)$ is a *cone* for a sequence of vertices $(u_1, \dots, u_n) \in V(D)^n$ if $c \sim u_i$ for all i . A vertex $c \in V(D)$ is a cone for the sequence of vertices (u_1, \dots, u_n) of *type* $(t_1, \dots, t_n) \in M^n$ if the following holds for every i :

- if $t_i = \rightarrow$ then $c \rightarrow u_i$,
- if $t_i = \leftarrow$ then $u_i \rightarrow c$, and
- if $t_i = \leftrightarrow$ then $c \leftrightarrow u_i$.

We say that a cone c of type (t_1, \dots, t_n) for some sequence of vertices is *not weaker than* the cone c' of type (t'_1, \dots, t'_n) for some (other) sequence of vertices if $(t'_1, \dots, t'_n) \leq (t_1, \dots, t_n)$. We write $c' \preceq c$. The proof of the following lemma is analogous to the proof of [5, Theorem 6]:

Lemma 3.1 *A reflexive improper digraph D is not homomorphism-homogeneous if and only if there are vertices $u_1, \dots, u_m, w_1, \dots, w_m \in V(D)$ and a homomorphism $f : D[u_1, \dots, u_m] \rightarrow D[w_1, \dots, w_m] : u_i \mapsto w_i$ with the following property:*

- either (u_1, \dots, u_m) has a cone and (w_1, \dots, w_m) does not,
- or (u_1, \dots, u_m) has a cone c such that $c \not\preceq d$ for every cone d of (w_1, \dots, w_m) .

Theorem 3.2 *The problem of deciding whether an improper finite reflexive bidirectionally connected digraph is homomorphism-homogeneous is coNP-complete.*

Proof. Let us first show that the problem is in coNP having in mind the criterion of homomorphism-homogeneity provided by Lemma 3.1. Given an improper digraph D and a triple $((u_1, \dots, u_m), (w_1, \dots, w_m), f)$ where f is a homomorphism from $D[u_1, \dots, u_m]$ to $D[w_1, \dots, w_m]$ that takes u_i to w_i , one can check in polynomial time that

- either (u_1, \dots, u_m) has a cone and (w_1, \dots, w_m) does not,
- or (u_1, \dots, u_m) has a cone c such that $c \not\preceq d$ for every cone d of (w_1, \dots, w_m) .

We prove the hardness by reducing the INDEPENDENT SET problem. Take any integer $k \geq 2$ and an irreflexive graph $G = (V, E)$ where $V = \{v_1, \dots, v_n\}$, and choose two $(k+1)$ -element sets $I = \{q_0, q_1, \dots, q_k\}$ and $S = \{s_0, s_1, \dots, s_k\}$ in such a way that V , I and S are pairwise disjoint. Let G_k be the reflexive improper digraph constructed as follows:

$$\begin{aligned}
V(G_k) &= V \cup I \cup S \\
E(G_k) &= E \cup \{v_i \rightarrow v_j : v_i \not\sim v_j \text{ in } G \text{ and } i < j\} \cup \{\text{all loops on } V \cup I \cup S\} \\
&\quad \cup \{s_i \rightleftharpoons s_j : i \neq j\} \cup \{q_i \rightarrow q_j : i < j\} \\
&\quad \cup \{v \rightleftharpoons q_j : v \in V, j > 0\} \cup \{v \rightarrow q_0 : v \in V\} \\
&\quad \cup \{v \rightleftharpoons s_j : v \in V, 0 \leq j \leq k\} \\
&\quad \cup \{s_i \rightleftharpoons q_j : i \neq j\} \cup \{s_i \rightarrow q_i : 0 \leq i \leq k\}.
\end{aligned}$$

Note the following:

- G is a graph, so if $v_i \sim v_j$ in G then $v_i \rightleftharpoons v_j$ in G_k ;
- for every pair of distinct vertices $x, y \in V(G_k)$ we have either $x \rightarrow y$ or $y \rightarrow x$ or $x \rightleftharpoons y$;
- if X is an m -independent set in G , then $G_k[X]$ is a transitive tournament on m vertices (modulo loops);
- $G_k[I]$ is a transitive tournament on $k+1$ vertices (modulo loops).

Let us show that G has a k -independent set if and only if G_k is not homomorphism-homogeneous.

(\Rightarrow) Assume that $\{x_0, x_1, \dots, x_{k-1}\} \subseteq V$ is a k -independent set in G . Then $G_k[x_0, x_1, \dots, x_{k-1}]$ is a transitive tournament (modulo loops, of course) and without loss of generality we can assume that $x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-1}$. The mapping

$$f = \begin{pmatrix} x_0 & x_1 & \dots & x_{k-1} & q_0 \\ q_0 & q_1 & \dots & q_{k-1} & q_k \end{pmatrix}.$$

is a homomorphism from $G_k[x_0, x_1, \dots, x_{k-1}, q_0]$ to $G_k[q_0, q_1, \dots, q_k]$. If G_k were homomorphism-homogeneous, then f would extend to an endomorphism f^* of G_k , so $f^*(s_1)$ would be a cone for (q_0, q_1, \dots, q_k) of type $(\rightrightarrows, \rightrightarrows, \dots, \rightrightarrows)$ since s_1 is a cone for $(x_0, x_1, \dots, x_{k-1}, q_0)$ of the same type. But it is easy to see that (q_0, q_1, \dots, q_k) does not have a cone of that type in G_k .

(\Leftarrow) Assume that G does not have a k -independent set and let us show that G_k is homomorphism-homogeneous. Clearly, it suffices to show that every homomorphism $f : G_k[U] \rightarrow G_k[W]$ where $W = f(U)$ can be extended to a homomorphism $f' : G_k[U \cup \{x\}] \rightarrow G_k[W \cup \{y\}]$ where $x \notin U$ and $f'(x) = y$.

Take any homomorphism $f : G_k[U] \rightarrow G_k[W]$ where $W = f(U)$ and $U \neq V \cup S \cup I$. If $I \not\subseteq W$, say, $q_i \notin W$ for some $q_i \in I$, then s_i is a cone for W in G_k of type $(\rightrightarrows, \rightrightarrows, \dots, \rightrightarrows)$. Now take any $x \notin U$ and note that $f' : G_k[U \cup \{x\}] \rightarrow G_k[W \cup \{s_i\}]$ where $f'(x) = s_i$ and $f'(y) = f(y)$ for $y \neq x$ is a homomorphism which extends f .

Assume, now, that $I \subseteq W$. Then there exist $x_0, \dots, x_k \in U$ such that $f(x_i) = q_i$, $0 \leq i \leq k$. Since $G_k[I]$ is a transitive tournament on $k+1$ vertices (modulo loops), so is $G_k[X]$ where $X = \{x_0, \dots, x_k\}$. Let us show that $X \cap S = \emptyset$. Clearly, $|X \cap S| \leq 1$ since every pair of distinct vertices from S is connected by a double edge, while $G_k[X]$ is a tournament. If $|X \cap S| = 1$, say, $X \cap S = \{s_i\}$, then $X \cap V = \emptyset$ since every vertex from S is connected by a double edge to every vertex from V . Therefore, $|X \cap I| = k \geq 2$, so there exists a $j \neq i$ such that $q_j \in X \cap I$. But, $q_j \rightrightarrows s_i$ by construction, which contradicts the fact that $G_k[X]$ is a tournament. This shows that $X \cap S = \emptyset$.

Next, let us show that $X \cap V = \emptyset$. Assume this is not the case and let $v \in X \cap V$. Since G does not have a k -independent set, it follows that no k -element subset of V induces a tournament in G_k . So, $|X \cap V| \leq k-1$, whence $|X \cap I| \geq 2$. Consequently, there exists an $i > 0$ such that $q_i \in X$. But, $q_i \rightrightarrows v$ by construction, which contradicts the fact that $G_k[X]$ is a tournament. This shows that $X \cap V = \emptyset$.

Therefore, $X = I$ so $f(q_i) = q_i$, $0 \leq i \leq k$, since $G_k[I]$ is a transitive tournament. Moreover, the argument above shows that

$$\text{if } f(x) \in I \text{ then } x \in I. \quad (\star)$$

If $V \not\subseteq U$, take any $v \in V \setminus U$ and extend f by setting $f'(v) = s_0$ and $f'(y) = f(y)$ for $y \in U$. Then f' is a homomorphism (which clearly extends f) since $v \rightarrow q_0$ and $s_0 \rightarrow q_0$ by construction, while $s_0 \rightrightarrows x$ for all

$x \neq q_0$. If, however, $V \subseteq U$, then $S \not\subseteq U$. Take any $s_i \in S \setminus U$ and extend f by setting $f'(s_i) = s_i$ and $f'(y) = f(y)$ for $y \in U$. It is easy to see that

$$f' = \begin{pmatrix} \overbrace{q_0 \dots q_k}^I & \overbrace{v_1 \dots v_n}^V & \overbrace{s_{j_1} \dots s_{j_t}}^{U \cap S} & s_i \\ q_0 \dots q_k & z_1 \dots z_n & w_1 \dots w_t & s_i \end{pmatrix}$$

is a homomorphism from $G_k[U \cup \{s_i\}]$ to $G_k[W \cup \{s_i\}]$: $s_i \rightleftharpoons x$ for all $x \neq q_i$, and $q_i \notin \{z_1, \dots, z_n, w_1, \dots, w_t\}$ because of (\star) . \square

4. Digraphs with involution

The classification of bidirectionally disconnected systems heavily relies on the following peculiar class of digraphs. Let D be a reflexive improper digraph. We say that D is a *digraph with involution* if there exists an automorphism $'$ of D satisfying

(DI1) $x = x''$;

(DI2) if $x \rightarrow y$ then $y \rightarrow x'$;

(DI3) if x and y are distinct vertices satisfying $x \rightleftharpoons y$ then $y = x'$.

Lemma 4.1 *Let D be a digraph with involution $'$. Then, for all $x, y \in V(D)$,*

- (a) $x \rightleftharpoons x'$;
- (b) $x = x'$ if and only if x is an isolated vertex of D ;
- (c) if $x \neq y$ and $x \sim y$ then $\{x, x'\} \bowtie \{y, y'\}$.

Proof. (a) Since D is reflexive, we have $x \rightarrow x$ and $x' \rightarrow x'$. From (DI2) we now conclude $x \rightarrow x'$ and $x' \rightarrow x'' = x$.

(b) Assume that x is an isolated vertex of D . Then by (a) we have $x \rightleftharpoons x'$ whence $x = x'$. For the converse, assume that $x = x'$ and let us show that x is then an isolated vertex of D . Suppose this is not the case, and let y be a vertex distinct from x such that $x \sim y$, say $x \rightarrow y$. Then by (DI2) we conclude that $y \rightarrow x' = x$. Therefore, $x \rightleftharpoons y$, so (DI3) now yields $y = x' = x$, which contradicts the assumption $y \neq x$.

(c) Assume that $x \rightarrow y$. Then $y \rightarrow x'$ by (DI2), $x' \rightarrow y'$ since $'$ is an automorphism of D and, by the same argument, $y' \rightarrow x'' = x$. \square

Clearly, if D is a digraph with involution $'$ then each class S of $\theta(D)$ takes the form $\{x, x'\}$ (see Fig. 2 (a)). So we have the following:

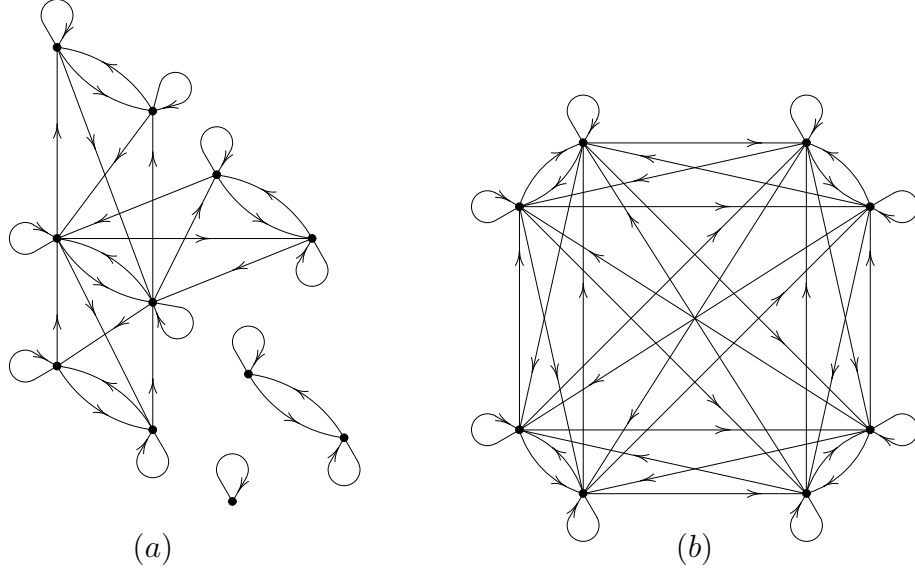


Figure 2: (a) A digraph with involution; (b) A tournament with involution

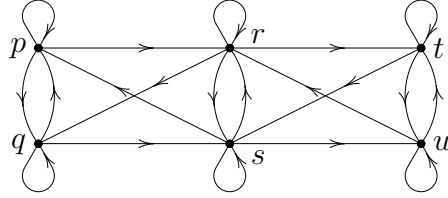


Figure 3: The proof of Lemma 4.3

Corollary 4.2 *If D is a digraph with involution, then the automorphism $'$ of D satisfying (DI1), (DI2) and (DI3) is unique.*

A digraph with involution is a *tournament with involution* if $x \sim y$ for all $x, y \in V(D)$ (Fig. 2 (b)).

Lemma 4.3 *Let D be a homomorphism-homogeneous digraph with involution. Then, for every connected component S of D , we have that $D[S]$ is a tournament with involution.*

Proof. Suppose that there exists a connected component of D such that $D[S]$ is not a tournament with involution. Then there exist $\{p, q\}, \{r, s\}, \{t, u\} \in V(D)/\theta(D)$ such that $\{p, q\} \bowtie \{r, s\} \bowtie \{t, u\}$, but $\neg(\{p, q\} \bowtie \{t, u\})$. Hence,

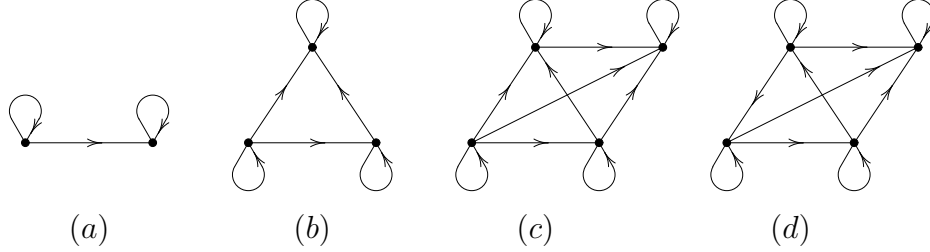


Figure 4: Bases of α_2 , α_3 , α_4 and ζ_4

$\{p, q\} \not\sim \{t, u\}$. Without loss of generality we can assume that $p \rightarrow r \rightarrow q \rightarrow s \rightarrow p$ and $r \rightarrow t \rightarrow s \rightarrow u \rightarrow r$, Fig. 3. The mapping

$$f : \begin{pmatrix} q & t & u \\ q & t & t \end{pmatrix}$$

is a homomorphism between finitely induced substructures of D so it extends to an endomorphism f^* of D . From $u \rightarrow r \rightarrow t$ it follows that $f^*(r) \rightleftharpoons t$, so $f^*(r) \in \{u, t\}$. On the other hand, $r \rightarrow q$ implies $f^*(r) \rightarrow q$. Therefore, $\{t, u\} \sim \{p, q\}$. Contradiction. \square

Let D be a tournament with involution such that $|V(D)| \geq 2$. Let S_1, \dots, S_k be the $\theta(D)$ -classes of D . Recall that each S_i takes the form $\{x, x'\}$ for some x . Take arbitrary $x_1 \in S_1$. Then in each S_j , $j \geq 2$, one of the vertices dominates x_1 while the other vertex is dominated by x_1 . For each $j \in \{2, \dots, k\}$ let $x_j \in S_j$ be the vertex which dominates x_1 . Clearly, $D[x_1, \dots, x_k]$ is a reflexive tournament which, up to isomorphism, uniquely determines D . We shall say that $D[x_1, \dots, x_k]$ is a *base* of D and write $D[x_1 \rightleftharpoons x_2, \dots, x_k]$ to emphasize the special status of x_1 . We say that a tournament with involution is *acyclic* if each of its bases is an acyclic reflexive tournament. Let α_n denote the acyclic tournament with involution with $2n$ vertices, $n \geq 1$, and let α_0 be the trivial one-vertex tournament with involution $\mathbf{1}^\circ$. The bases of α_2 , α_3 and α_4 are depicted in Fig. 4 (a), (b) and (c), respectively. Let ζ_4 denote the tournament with involution with 8 vertices whose base is depicted in Fig. 4 (d). Up to isomorphism, there are four distinct tournaments with involution with 4, 6 and 8 vertices: α_2 , α_3 , α_4 and ζ_4 , and one can easily check that all of them are homomorphism-homogeneous.

Lemma 4.4 *Let D_1 and D_2 be tournaments with involution and let $f : U \rightarrow W$ be a homomorphism from $D_1[U]$ to $D_2[W]$. Assume that there is a $u \in U$ such that $u' \in U$ and $f(u) = f(u')$. Then*

- (a) $f(U) \subseteq \{v, v'\}$, where $v = f(u) = f(u')$;
- (b) f extends to a homomorphism f^* from D_1 to D_2 .

Proof. (a) Take any $x \in U \setminus \{u, u'\}$. Since D_1 is a tournament with involution, we have $x \rightarrow u$ or $u \rightarrow x$, but not both. Say, $x \rightarrow u$. Then $u' \rightarrow x \rightarrow u$ whence $v \rightleftharpoons f(x)$. Therefore, $f(x) \in \{v, v'\}$, since D_2 is also a tournament with involution.

(b) It is easy to see that $f^* : V(D_1) \rightarrow V(D_2)$ defined by $f^*(x) = x$ if $x \in U$ and $f^*(x) = v$ if $x \notin U$ is a homomorphism from D_1 to D_2 . \square

Lemma 4.5 *Let D_1 and D_2 be tournaments with involution and let $f : U \rightarrow W$ be a homomorphism from $D_1[U]$ to $D_2[W]$. Assume that $f(U) \not\subseteq \{v, v'\}$ for all $v \in V(D_2)$. Then $f(u') = f(u)'$ whenever $u, u' \in U$.*

Proof. Assume that $u, u' \in U$. Since $u \rightleftharpoons u'$, we have that $f(u) \rightleftharpoons f(u')$, so $f(u') \in \{f(u), f(u)'\}$. If $f(u') = f(u)$ then, as we have just seen in Lemma 4.4, $f(U) \subseteq \{v, v'\}$ for $v = f(u)$, which is not the case. Therefore $f(u') = f(u)'$. \square

Lemma 4.6 *Let D_1 and D_2 be tournaments with involution and let $f : U \rightarrow W$ be a homomorphism from $D_1[U]$ to $D_2[W]$ such that $f(u') = f(u)'$ whenever $u, u' \in U$. Then f extends to a homomorphism \bar{f} from $D_1[\bar{U}]$ to $D_2[\bar{W}]$ where $\bar{U} = U \cup \{u' : u \in U\}$ and $\bar{W} = W \cup \{w' : w \in W\}$.*

Proof. Assume that there is an $x \in U$ such that $x' \notin U$. Then $f_1 : U \cup \{x\} \rightarrow W \cup \{f(x)'\}$ defined by

$$f_1(u) = \begin{cases} f(u), & u \in U \\ f(x)', & u = x' \end{cases}$$

is a homomorphism from $D_1[U \cup \{x\}]$ to $D_2[W \cup \{f(x)'\}]$. We can repeat this procedure for every $x \in U$ such that $x' \notin U$ and thus extend f to \bar{U} . \square

Lemma 4.7 $\alpha_m \Rightarrow \alpha_n$ for all $m, n \geq 0$.

Proof. If $m \leq 1$ or $n \leq 1$ the claim is trivially true. Assume, therefore, that $m, n \geq 2$. Fix a base $\alpha_m[x_1 \leftarrow x_2, \dots, x_m]$ of α_m . Let $f : U \rightarrow W$ be a homomorphism from $\alpha_m[U]$ to $\alpha_n[W]$ where $U = \{u_1, \dots, u_l\}$ and $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_l$. Due to Lemmas 4.4 and 4.5 without loss of generality we can assume that $f(u') = f(u)'$ whenever $u, u' \in U$.

Let $\overline{U} = U \cup \{x_1, \dots, x_m\}$ and let $\overline{f} : \overline{U} \rightarrow W$ be the mapping defined by $\overline{f}(u) = f(u)$ for $u \in U$ and $\overline{f}(x_i) = f(u_{\alpha(i)})$, $1 \leq i \leq m$, where

$$\alpha(i) = \begin{cases} \min\{k : x_i \rightarrow u_k\}, & x_i \rightarrow u_l \\ l, & x_i \not\rightarrow u_l. \end{cases}$$

Then \overline{f} is well-defined (if $x_i \in U$, say $x_i = u_j$ for some j , then $\overline{f}(x_i) = f(u_{\alpha(i)}) = f(u_j) = f(x_i)$) and it is a homomorphism from $\alpha_m[\overline{U}]$ to $\alpha_n[W]$ since $x_i \rightarrow x_j$ implies $u_{\alpha(i)} \rightarrow u_{\alpha(j)}$. According to Lemma 4.6, \overline{f} now easily extends to a homomorphism $f^* : \alpha_m \rightarrow \alpha_n$. \square

Theorem 4.8 Let D be a tournament with involution. Then D is homomorphism-homogeneous if and only if

- (1) $D \cong \zeta_4$, or
- (2) $D \cong \alpha_n$ for some $n \geq 0$.

Proof. (\Leftarrow) We have already seen that ζ_4 is homomorphism-homogeneous. From Lemma 4.7 it follows that $\alpha_n \Rightarrow \alpha_n$ for all $n \geq 1$, so α_n is homomorphism-homogeneous for all $n \geq 0$.

(\Rightarrow) Let D be a homomorphism-homogeneous tournament with involution. If $|V(D)| \leq 8$ then $D \cong \zeta_4$ or $D \cong \alpha_n$ for some $n \in \{0, 1, 2, 3, 4\}$. Assume now that $|V(D)| \geq 10$ and let us show that D is an acyclic tournament with involution.

Suppose, to the contrary, that D is not an acyclic tournament with involution and let $D[x_1 \leftarrow x_2, \dots, x_k]$ be a base of D which is not an acyclic tournament. Then $D[x_2, \dots, x_k]$ is not acyclic. Since $|V(D)| \geq 10$, every base of D has at least 5 vertices, so $k \geq 5$.

Assume that there exist distinct $y_1, \dots, y_m \in \{x_2, \dots, x_k\}$ such that $y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_m \rightarrow y_1$ is a cycle of length $m \geq 4$. Then $D[y_1, \dots, y_m]$ is a Hamiltonian tournament and hence pancyclic. Therefore, there exist four

distinct vertices $p, q, r, s \in \{y_1, \dots, y_m\}$ such that $p \rightarrow q \rightarrow r \rightarrow s \rightarrow p$ is a 4-cycle, Fig. 5 (a) (instead of p and q the figure depicts p' and q'). The mapping

$$f = \begin{pmatrix} p' & q & r & s' \\ r & q & r & q \end{pmatrix}$$

is a homomorphism from $D[p', q, r, s']$ to $D[q, r]$, so it extends to an endomorphism f^* of D . From $r \rightarrow x_1 \rightarrow p'$ we have $f^*(r) \rightarrow f^*(x_1) \rightarrow f^*(p')$, that is, $r \rightarrow f^*(x_1) \rightarrow r$. Therefore, $f^*(x_1) \in \{r, r'\}$. Analogously, from $q \rightarrow x_1 \rightarrow s'$ we conclude $f^*(x_1) \in \{q, q'\}$. Contradiction.

Assume now that there exist distinct $p, q, r \in \{x_2, \dots, x_k\}$ such that $p \rightarrow q \rightarrow r \rightarrow p$ is a 3-cycle. Since $k \geq 5$ there exists an $s \in \{x_2, \dots, x_k\} \setminus \{p, q, r\}$. As we have just seen, $D[x_2, \dots, x_k]$ does not contain a 4-cycle, so $\{p, q, r\} \rightrightarrows s$ or $s \rightrightarrows \{p, q, r\}$. Without loss of generality we can assume that $\{p, q, r\} \rightrightarrows s$, Fig. 5 (b). The mapping

$$f = \begin{pmatrix} p & q & s & x_1 \\ p & q & q & x_1 \end{pmatrix}$$

is a homomorphism from $D[p, q, s, x_1]$ to $D[p, q, x_1]$, so it extends to an endomorphism f^* of D . From $q \rightarrow r \rightarrow s$ we conclude that $q \rightarrow f^*(r) \rightarrow q$, so $f^*(r) \in \{q, q'\}$. If $f^*(r) = q$ then $r \rightarrow p$ implies $q \rightarrow p$, which is not the case. On the other hand, if $f^*(r) = q'$ then $r \rightarrow x_1$ implies $q' \rightarrow x_1$, which is not possible (since $q \rightarrow x_1$ enforces $x_1 \rightarrow q'$). Therefore, $D[x_2, \dots, x_k]$ is acyclic, and hence $D[x_1 \leftarrow x_2, \dots, x_k]$ is acyclic. \square

Lemma 4.9 *For all $n \geq 2$ we have $\zeta_4 \not\equiv \alpha_n$.*

Proof. Assume first that $\zeta_4 \Rightarrow \alpha_2$. Let $\zeta_4[s \leftarrow p, q, r]$ be a base of ζ_4 such that $p \rightarrow q \rightarrow r \rightarrow p$, and let $\alpha_2[t \leftarrow u]$ be a base of α_2 , Fig. 6. The mapping

$$f = \begin{pmatrix} p & s & q \\ u & t & t \end{pmatrix}$$

is a homomorphism from $\zeta_4[s, p, q]$ to $\alpha_2[t, u]$, so by the assumption it extends to a homomorphism f^* from ζ_4 to α_2 . Let us compute $f^*(r)$. From $r \rightarrow s$ it follows that $f^*(r) \rightarrow t$. Therefore, $f^*(r)$ is a vertex in the base of α_2 under consideration, so $f^*(r) \in \{u, t\}$. If $f^*(r) = u$ then $q \rightarrow r$ implies $f^*(q) = t \rightarrow u = f^*(r)$, which is not the case. On the other hand, if

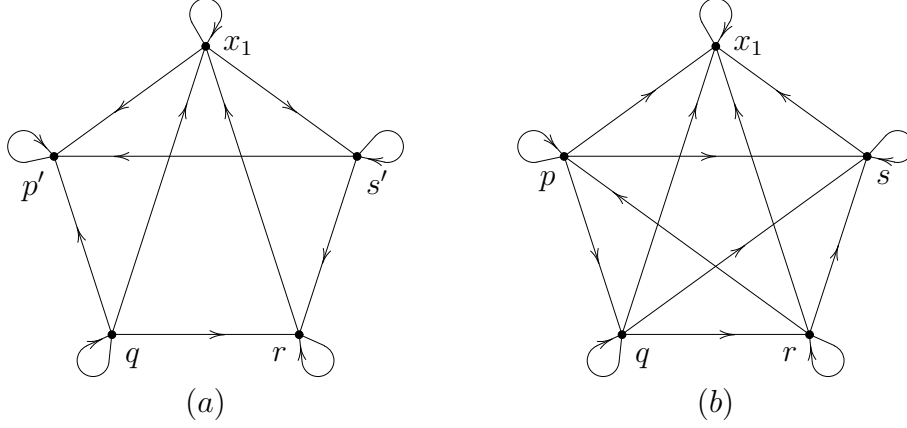


Figure 5: The proof of Theorem 4.8

$f^*(r) = t$ then $r \rightarrow p$ implies $f^*(r) = t \rightarrow u = f^*(p)$, which is not the case. This contradiction shows that $\zeta_4 \not\Rightarrow \alpha_2$.

Assume now that $\zeta_4 \Rightarrow \alpha_n$ for some $n \geq 3$. As above, let $\zeta_4[s \Leftarrow p, q, r]$ be a base of ζ_4 such that $p \rightarrow q \rightarrow r \rightarrow p$, and let $\alpha_n[t \Leftarrow u, v, x_4, \dots, x_n]$ be a base of α_n such that $u \rightarrow v$. The mapping

$$f = \begin{pmatrix} p & s & q \\ u & t & v \end{pmatrix}$$

is a homomorphism from $\zeta_4[s, p, q]$ to $\alpha_n[t, u, v]$, so by the assumption it extends to a homomorphism f^* from ζ_4 to α_n . Let us compute $f^*(r)$. From $r \rightarrow s$ it follows that $f^*(r) \rightarrow t$. Therefore, $f^*(r)$ is a vertex in the base of α_n under consideration, that is, $f^*(r) \in \{t, u, v, x_4, \dots, x_n\}$. If $f^*(r) = t$ then $r \rightarrow p$ implies $f^*(r) = t \rightarrow u = f^*(p)$, which is not the case. If $f^*(r) = u$ then $q \rightarrow r$ implies $f^*(q) = v \rightarrow u = f^*(p)$, which is not the case. If $f^*(r) = v$ then $r \rightarrow p$ implies $f^*(r) = v \rightarrow u = f^*(p)$, which is not the case. Finally, if $f^*(r) = x_i$ for some i then $q \rightarrow r \rightarrow p$ implies $v \rightarrow x_i \rightarrow u$, which is not the case since $\alpha_n[t \Leftarrow u, v, x_4, \dots, x_n]$ is an acyclic digraph. Therefore, $\zeta_4 \not\Rightarrow \alpha_n$. \square

Theorem 4.10 *Let D be a digraph with involution. Then D is homomorphism-homogeneous if and only if*

- (1) $D \cong k \cdot \alpha_0 + l \cdot \alpha_1 + m \cdot \zeta_4$ for some $k, l, m \geq 0$, or

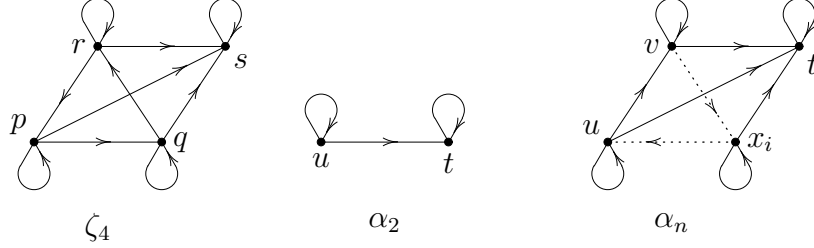


Figure 6: The proof of Lemma 4.9

- (2) $D \cong m_1 \cdot \alpha_{n_1} + \dots + m_k \cdot \alpha_{n_k}$ for some $k \geq 1$ and $m_1, \dots, m_k, n_1, \dots, n_k \geq 0$.

Proof. (\Leftarrow) It is easy to see that $D_1 \Rightarrow D_2$ for all choices of $D_1, D_2 \in \{\alpha_0, \alpha_1, \zeta_4\}$, so digraphs from the class (1) are homomorphism-homogeneous. We have seen in Lemma 4.7 that $\alpha_{n_i} \Rightarrow \alpha_{n_j}$ for all $n_i, n_j \geq 0$, so digraphs from the class (2) are also homomorphism-homogeneous.

(\Rightarrow) Let D be a homomorphism-homogeneous digraph with involution. Then by Lemma 4.3 every connected component of D is a homomorphism-homogeneous tournament with involution. Therefore, Theorem 4.8 yields that every connected component of D is isomorphic to ζ_4 or α_n for some $n \geq 0$. If there is a connected component of D isomorphic to ζ_4 , then due to Lemma 4.9 every connected component of D is isomorphic to ζ_4 , α_0 or α_1 , and we have case (1). On the other hand, if no connected component of D is isomorphic to ζ_4 then every connected component of D is isomorphic to α_n for some $n \geq 0$ and we have case (2). \square

5. Bidirectionally disconnected systems

Reflexive homomorphism-homogeneous proper digraphs were characterized in [4, Theorem 3.10], see Theorem 2.4. As for bidirectionally connected digraphs, we have seen in Theorem 3.2 that we cannot hope for a reasonable description due to the complexity of the corresponding decision problem. In this section we characterize finite reflexive homomorphism-homogeneous bidirectionally disconnected improper digraphs.

Let us start with a rather general result. We say that a digraph (V^*, E^*) is a *retract* of a digraph (V, E) if there exist homomorphisms $r : V \rightarrow V^*$ and $j : V^* \rightarrow V$ such that $r \circ j = \text{id}_{V^*}$.

Lemma 5.1 *Let D be a reflexive improper digraph and let $\rho \subseteq V(D)^2$ be an equivalence relation on $V(D)$ such that the following holds:*

- *for every $S \in V(D)/\rho$ we have $D[S] \cong K_n^\circ$ for some positive integer n ;*
- *for all distinct $S, T \in V(D)/\rho$, if $S \sim T$ then $S \rightrightarrows T$ or $T \rightrightarrows S$ or both.*

Define the digraph D/ρ as follows: the set of vertices of D/ρ is $V(D)/\rho$, while (S, T) is an edge of D/ρ if and only if $S = T$ or $S \rightrightarrows T$ in D . Then

- (1) *D/ρ is a reflexive digraph.*
- (2) *D/ρ is a retract of D .*
- (3) *If D is a homomorphism-homogeneous digraph then D/ρ is a homomorphism-homogeneous digraph.*
- (4) *Assume that the following holds for D :*
 (\diamond) *for all distinct $S, T \in V(D)/\rho$, if $S \sim T$ then $S \rightrightarrows T$ or $T \rightrightarrows S$, but not both.*

Then D is a homomorphism-homogeneous digraph if and only if D/ρ is a homomorphism-homogeneous digraph.

Proof. Let $V(D)/\rho = \{S_1, \dots, S_n\}$.

(1) Obvious.

(2) Choose arbitrary $s_1 \in S_1, \dots, s_n \in S_n$ and define $r : V(D) \rightarrow V(D/\rho)$ and $j : V(D/\rho) \rightarrow V(D)$ by $r(x) = x/\rho$ and $j(S_i) = s_i$. Then r and j are homomorphisms satisfying $r \circ j = \text{id}_{V(D/\rho)}$, so D/ρ is a retract of D .

(3) It is easy to see that every retract of a homomorphism-homogeneous relational structure is homomorphism-homogeneous. Therefore, D/ρ , being a retract of D , is homomorphism-homogeneous.

(4) Direction from left to right follows from (3). Let us show the other direction. Let $f : U \rightarrow W$ be a homomorphism from $D[U]$ to $D[W]$ where $U, W \subseteq V(D)$. From (\diamond) it follows that if $U \cap S \neq \emptyset$ for some $S \in V(D)/\rho$ then $f(U \cap S) \subseteq S'$ for some $S' \in V(D)/\rho$. Without loss of generality, let S_1, \dots, S_k be all the ρ -classes that intersect U and let S'_1, \dots, S'_k be the ρ -classes such that $f(U \cap S_i) \subseteq S'_i$, $1 \leq i \leq k$. Then the mapping $g : \{S_1, \dots, S_k\} \rightarrow \{S'_1, \dots, S'_k\} : S_i \mapsto S'_i$ is easily seen to be a homomorphism from $(D/\rho)[S_1, \dots, S_k]$ to $(D/\rho)[S'_1, \dots, S'_k]$. Since D/ρ is homomorphism-homogeneous, g extends to an endomorphism g^* of D/ρ . Then the mapping

$f^* : V(D) \rightarrow V(D)$ defined by

$$f^*(x) = \begin{cases} f(x), & x \in U \\ j \circ g^* \circ r(x), & x \notin U \end{cases}$$

where r and j are the homomorphisms from (2), is an endomorphism of D which extends f . \square

Recall that a quasiorder is a binary relational system (A, \leq) where \leq is a reflexive and transitive binary relation on A . If \equiv denotes the equivalence relation on A defined by $x \equiv y$ if $x \leq y$ and $y \leq x$, then A/\equiv is a partially ordered set where $x/\equiv \leq y/\equiv$ if and only if $x \leq y$. Since $(A/\equiv, \leq)$ is a retract of (A, \leq) , as a direct consequence of the above lemma we have the following:

Corollary 5.2 *Let (A, \leq) be a quasiorder. Then (A, \leq) is homomorphism-homogeneous as a quasiorder if and only if $(A/\equiv, \leq)$ is a homomorphism-homogeneous partially ordered set.*

Let $D = (V, E)$ be a proper digraph with $V = \{v_1, \dots, v_n\}$, and let V_1, \dots, V_n be finite nonempty pairwise disjoint sets. Let $D\langle V_1, \dots, V_n \rangle$ denote the digraph whose vertices are $V_1 \cup \dots \cup V_n$ and whose edges are defined as follows:

- for every $i \in \{1, \dots, n\}$ and for all $x, y \in V_i$ we have $x \rightarrow y$ in $D\langle V_1, \dots, V_n \rangle$;
- if $v_i \rightarrow v_j$ in D and $i \neq j$, then $V_i \rightrightarrows V_j$ in $D\langle V_1, \dots, V_n \rangle$;
- no other edges exist in $D\langle V_1, \dots, V_n \rangle$.

We say that $D\langle V_1, \dots, V_n \rangle$ is an *inflation* of D . Note that $D\langle V_1, \dots, V_n \rangle[V_i] \cong K_{|V_i|}^\circ$ and that D is a retract of $D\langle V_1, \dots, V_n \rangle$.

Lemma 5.3 *A finite reflexive proper digraph D is homomorphism-homogeneous if and only if every inflation of D is homomorphism-homogeneous.*

Lemma 5.4 *Let D be a finite homomorphism-homogeneous reflexive bidirectionally disconnected improper digraph, and let $S \in V(D)/\theta(D)$ be an arbitrary equivalence class of $\theta(D)$. Then $D[S] \cong K_n^\circ$ for some positive integer n .*

Proof. From $\omega(D) < |V(D)/\theta(D)|$ it follows that there exist distinct classes of $\theta(D)$ which belong to the same connected component of D . Therefore, we can choose $T_1, T_2 \in V(D)/\theta(D)$ in such a way that $T_1 \neq T_2$ and $T_1 \rightarrow T_2$. Moreover, choose an $y_1 \in T_1$ and an $y_2 \in T_2$ so that $y_1 \rightarrow y_2$.

Assume that there is an $S \in V(D)/\theta(D)$ such that $D[S]$ is not a complete reflexive graph. Then there exist $u, v \in S$ such that $u \not\Rightarrow v$. If $u \not\sim v$ or $u \rightarrow v$, consider the mapping

$$f : \begin{pmatrix} u & v \\ y_1 & y_2 \end{pmatrix}.$$

If $v \rightarrow u$, consider

$$f : \begin{pmatrix} v & u \\ y_1 & y_2 \end{pmatrix}.$$

In any case, the mapping f is a homomorphism between finitely induced subdigraphs of D , so it extends to an endomorphism f^* of D . Since u and v belong to the same equivalence class of $\theta(D)$, there exist $z_1, z_2, \dots, z_k \in V(D)$ such that

$$u = z_1 \rightleftharpoons z_2 \rightleftharpoons \dots \rightleftharpoons z_k = v.$$

Then $f^*(u) = f^*(z_1) \rightleftharpoons f^*(z_2) \rightleftharpoons \dots \rightleftharpoons f^*(z_k) = f^*(v)$, whence follows that $(y_1, y_2) \in \theta(D)$ since $\{f^*(u), f^*(v)\} = \{y_1, y_2\}$. Contradiction. \square

Bidirectionally disconnected digraphs naturally split into two classes:

- we say that a digraph D is a digraph *with no back-and-forth* if the following holds for all $S, T \in V(D)/\theta(D)$: if $S \rightleftharpoons T$ then $S = T$;
- we say that a digraph D is a digraph *with back-and-forth* if there exist distinct $S, T \in V(D)/\theta(D)$ such that $S \rightleftharpoons T$.

Let us first classify homomorphism-homogeneous bidirectionally disconnected digraphs with no back-and-forth.

Lemma 5.5 *Let D be a finite homomorphism-homogeneous reflexive bidirectionally disconnected improper digraph with no back-and-forth. Then for all distinct $S, T \in V(D)/\theta(D)$ either $S \not\sim T$, or $S \Rightarrow T$ or $T \Rightarrow S$.*

Proof. Take any $S, T \in V(D)/\theta(D)$ such that $S \sim T$, assume that $S \rightarrow T$ and let us show that $S \Rightarrow T$. Assume, to the contrary, that there exist $v \in S$

and $w \in T$ such that $v \not\rightarrow w$. Since $S \neq T$ and $S \rightarrow T$, we know that $T \not\rightarrow S$, so $w \not\rightarrow v$ and thus $v \not\sim w$. Then the mapping

$$f : \begin{pmatrix} v & w \\ w & v \end{pmatrix}$$

is a homomorphism between finitely induced subdigraphs of D , so it extends to an endomorphism f^* of D . Choose $x \in S$ and $y \in T$ so that $x \rightarrow y$. From $v \rightarrow x \rightarrow y \rightarrow w$ it follows that $f^*(v) \rightarrow f^*(x) \rightarrow f^*(y) \rightarrow f^*(w)$, that is, $w \rightarrow f^*(x) \rightarrow f^*(y) \rightarrow v$. Clearly, at least one of the edges $w \rightarrow f^*(x)$, $f^*(x) \rightarrow f^*(y)$ or $f^*(y) \rightarrow v$ leads from T to S , which contradicts the fact that $T \not\rightarrow S$. \square

Theorem 5.6 *Let D be a finite reflexive bidirectionally disconnected improper digraph with no back-and-forth. Then D is homomorphism-homogeneous if and only if*

- (1) *D is a finite homomorphism-homogeneous quasiorder; or*
- (2) *D is an inflation of $k \cdot C_3^\circ + l \cdot \mathbf{1}^\circ$ for some $k, l \geq 0$ such that $k + l \geq 1$.*

Proof. Let D be a finite reflexive bidirectionally disconnected improper digraph with no back-and-forth.

(\Rightarrow) Assume that D is a homomorphism-homogeneous digraph. According to Lemma 5.1, $D/\theta(D)$ is a homomorphism-homogeneous reflexive proper digraph, so Theorem 2.4 yields that either $D/\theta(D) \cong k \cdot C_3^\circ + l \cdot \mathbf{1}^\circ$ for some $k, l \geq 0$ such that $k + l \geq 1$, or $D/\theta(D)$ is a finite homomorphism-homogeneous partially ordered set. Therefore, either D is an inflation of $k \cdot C_3^\circ + l \cdot \mathbf{1}^\circ$ for some $k, l \geq 0$ such that $k + l \geq 1$, or D is a finite homomorphism-homogeneous quasiorder (Corollary 5.2).

(\Leftarrow) Assume that D belongs to one of the classes (1)–(2). Then $D/\theta(D)$ is a homomorphism-homogeneous reflexive proper digraph according to Corollary 5.2 and Theorem 2.4. Lemma 5.1 now yields that D is a homomorphism-homogeneous improper digraph. \square

The classification of bidirectionally disconnected digraphs with back-and-forth is slightly more involved. Our intention is to prove that if D is a homomorphism-homogeneous bidirectionally disconnected digraph with back-and-forth, then $D/\theta(D)$ is a homomorphism-homogeneous digraph with involution.

Lemma 5.7 *Let D be a finite homomorphism-homogeneous reflexive bidirectionally disconnected improper digraph with back-and-forth. Then $S \sim T$ implies $S \rightleftharpoons T$ for all $S, T \in V(D)/\theta(D)$.*

Proof. Take $S, T \in V(D)/\theta(D)$ so that $S \sim T$ and assume that $s \rightarrow t$ for some $s \in S$ and $t \in T$. We know that there exist distinct $U, W \in V(D)/\theta(D)$ such that $U \rightleftharpoons W$, so choose $u_1, u_2 \in U$ and $w_1, w_2 \in W$ in such a way that $u_1 \rightarrow w_1$ and $w_2 \rightarrow u_2$. The mapping

$$f : \begin{pmatrix} u_1 & w_1 \\ s & t \end{pmatrix}$$

is a homomorphism between finitely induced subdigraphs of D , so it extends to an endomorphism f^* of D . It follows from Lemma 2.2 that $f^*(U) \subseteq S$ and $f^*(W) \subseteq T$, so $T \rightarrow S$ since $T \ni f^*(w_2) \rightarrow f^*(u_2) \in S$. Therefore, $S \rightleftharpoons T$. \square

Lemma 5.8 *Let D be a finite homomorphism-homogeneous reflexive bidirectionally disconnected improper digraph, and let S and T be distinct classes of $\theta(D)$ such that $S \rightleftharpoons T$.*

- (1) *There exists an $s \in S$ such that $s \rightleftharpoons T$, and a $t \in T$ such that $S \rightleftharpoons t$.*
- (2) *$|S| \geq 2$ and $|T| \geq 2$.*
- (3) *Suppose that $r \rightarrow t \rightarrow s$ for some $r, s \in S$ and $t \in T$. Then for every $u \in T$, either $r \rightarrow u \rightarrow s$ or $s \rightarrow u \rightarrow r$.*
- (4) *$s \sim t$ for all $s \in S$ and all $t \in T$.*

Proof. Let $S, T \in V(D)/\theta(D)$ be distinct classes of $\theta(D)$ such that $S \rightleftharpoons T$.

(1) Take $s_1, s_2 \in S$ and $t_1, t_2 \in T$ so that $s_1 \rightarrow t_1$ and $t_2 \rightarrow s_2$. It suffices to show that there exists a $s \in S$ such that $s \rightleftharpoons T$ **or** a $t \in T$ such that $S \rightleftharpoons t$, since the mapping

$$\begin{pmatrix} s_1 & t_1 \\ t_2 & s_2 \end{pmatrix}$$

extends to an endomorphism of D which then takes care of the other case.

If $t_1 \rightarrow s_2$ then $S \rightleftharpoons t_1$ and we are done. Assume now that $t_1 \not\rightarrow s_2$. Then the mapping

$$f : \begin{pmatrix} s_1 & s_2 & t_1 \\ s_1 & s_1 & t_1 \end{pmatrix}$$

is a homomorphism between finitely induced substructures of D and, by the homogeneity requirement, extends to an endomorphism f^* of D . From $t_2 \rightleftharpoons t_1$ it follows that $f^*(t_2) \rightleftharpoons t_1$, so $f^*(t_2) \in T$. Moreover, $t_2 \rightarrow s_2$ yields $f^*(t_2) \rightarrow s_1$. Therefore, $f^*(t_2) \rightarrow s_1 \rightarrow t_1$ and thus $s_1 \rightleftharpoons T$.

(2) Follows straightforwardly from (1) and the fact that S and T are disjoint classes of $\theta(D)$, so $s \not\rightleftharpoons t$ for all $s \in S$ and all $t \in T$.

(3) The statement trivially holds for t . Take any $u \in T \setminus \{t\}$ and let us show that the following two mappings cannot be homomorphisms between the corresponding induced substructures:

$$f : \begin{pmatrix} r & s & u \\ r & r & t \end{pmatrix} \quad \text{and} \quad g : \begin{pmatrix} s & r & u \\ s & s & t \end{pmatrix}.$$

Assume that f is a homomorphism from $D[r, s, u]$ to $D[r, t]$. Then f extends to an endomorphism f^* of D . Let us take a look at $f^*(t)$. From $u \rightleftharpoons t$ we infer $t = f^*(u) \rightleftharpoons f^*(t)$, so $f^*(t) \in T$. On the other hand, $r \rightarrow t \rightarrow s$ implies $r = f^*(r) \rightarrow f^*(t) \rightarrow f^*(s) = r$, that is $r \rightleftharpoons f^*(t)$, whence $f^*(t) \in S$. This contradicts the fact that S and T are disjoint. The proof for g is analogous.

Let us now show that $s \sim u$. Suppose, to the contrary, that $s \not\sim u$. If $u \not\rightarrow r$ then f above is a homomorphism between finitely induced substructures of D , which is impossible. If, however, $u \rightarrow r$ then g above is a homomorphism between finitely induced substructures of D , which is also impossible. Therefore, $s \sim u$.

If $s \rightarrow u$ then $u \rightarrow r$ (since $s \rightarrow u$ and $u \not\rightarrow r$ implies that f is a homomorphism between finitely induced substructures of D , which is impossible), and if $u \rightarrow s$ then $r \rightarrow u$ (since $u \rightarrow s$ and $r \not\rightarrow u$ implies that g is a homomorphism between finitely induced substructures of D , which is impossible).

(4) From (1) we know that there exist $q, r \in S$ and a $u \in T$ such that $q \rightarrow u \rightarrow r$. Take any $s \in S$ and any $t \in T$. Then (3) yields that $r \rightarrow t \rightarrow q$ or $q \rightarrow t \rightarrow r$. Clearly, if $s = r$ we are done, so we can assume that $s \neq r$.

Assume, first, that $r \rightarrow t \rightarrow q$, Fig. 7 (a). Then, clearly, $t \neq u$. Since $u \rightarrow r \rightarrow t$, then from (3) we know that either $u \rightarrow s \rightarrow t$ or $t \rightarrow s \rightarrow u$. Either way, $s \sim t$.

Assume, now, that $q \rightarrow t \rightarrow r$, Fig. 7 (b). The mapping

$$h : \begin{pmatrix} s & t \\ t & r \end{pmatrix}$$

is a homomorphism between finitely induced substructures of D , so it extends to an endomorphism h^* of D . Then $r \rightleftharpoons s$ implies $h^*(r) \rightleftharpoons t$, so $h^*(r) \in T$.

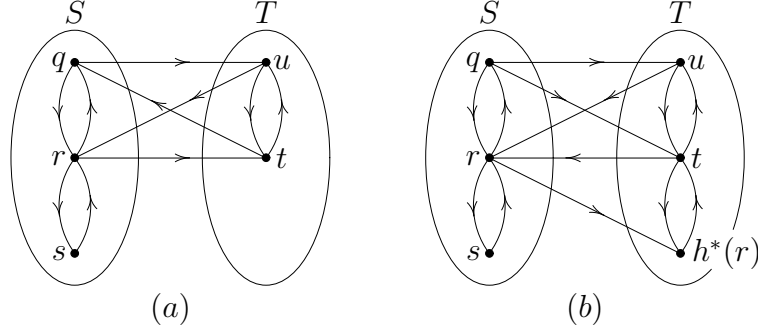


Figure 7: The proof of Lemma 5.8 (4)

Moreover, $t \rightarrow r$ implies $r \rightarrow h^*(r)$. Thus we get $t \rightarrow r \rightarrow h^*(r)$, so (3) ensures that $t \rightarrow s \rightarrow h^*(r)$ or $h^*(r) \rightarrow s \rightarrow t$. Either way, $s \sim t$. \square

Let S and T be distinct classes of $\theta(D)$ such that $S \rightleftharpoons T$. Define a binary relation $\gamma_T(S) \subseteq S^2$ on S as follows:

$$(x, y) \in \gamma_T(S) \quad \text{if and only if} \quad \neg \exists t \in T (x \rightarrow t \rightarrow y \vee y \rightarrow t \rightarrow x).$$

Lemma 5.9 *Let D be a finite homomorphism-homogeneous reflexive improper bidirectionally disconnected digraph, and let S and T be distinct classes of $\theta(D)$ such that $S \rightleftharpoons T$.*

- (1) $\gamma_T(S)$ is an equivalence relation on S .
- (2) $(x, y) \in \gamma_T(S)$ if and only if $\forall t \in T (x \rightarrow t \leftarrow y \vee x \leftarrow t \rightarrow y)$.
- (3) $(x, y) \in \gamma_T(S)$ if and only if $\exists t \in T (x \rightarrow t \leftarrow y \vee x \leftarrow t \rightarrow y)$.
- (4) $\gamma_T(S)$ has precisely two blocks.
- (5) Let $S/\gamma_T(S) = \{S_1, S_2\}$ and $T/\gamma_S(T) = \{T_1, T_2\}$. Then $S_1 \Rightarrow T_1 \Rightarrow S_2 \Rightarrow T_2 \Rightarrow S_1$ or $T_1 \Rightarrow S_1 \Rightarrow T_2 \Rightarrow S_2 \Rightarrow T_1$.

Proof. (1) The relation $\gamma_T(S)$ is obviously reflexive (because S and T are distinct classes of $\theta(D)$) and symmetric. Let us show that $\gamma_T(S)$ is transitive. Take any $(q, r), (r, s) \in S^2$ and assume that $(q, s) \notin \gamma_T(S)$. Then there exists a $t \in T$ such that $q \rightarrow t \rightarrow s$ or $s \rightarrow t \rightarrow q$. Without loss of generality we can assume that $q \rightarrow t \rightarrow s$. From Lemma 5.8 (4) we know that $x \sim y$ for all $x \in S$ and all $y \in T$, so $r \sim t$. Then $r \rightarrow t$ implies $(r, s) \notin \gamma_T(S)$, while $t \rightarrow r$ implies $(q, r) \notin \gamma_T(S)$. This shows that $\gamma_T(S)$ is an equivalence relation on S .

(2) Direction from right to left is obvious. In order to show the other direction, take any $(r, s) \in \gamma_T(S)$ and any $t \in T$. From Lemma 5.8 (4) we know that $x \sim y$ for all $x \in S$ and all $y \in T$, so $r \sim t \sim s$. Since $\neg(r \rightarrow t \rightarrow s)$ and $\neg(s \rightarrow t \rightarrow r)$, it must be the case that $r \rightarrow t \leftarrow s$ or $r \leftarrow t \rightarrow s$.

(3) Direction from left to right follows straightforwardly from (2). In order to show the other direction, take any $(r, s) \notin \gamma_T(S)$. Then there is a $t \in T$ satisfying $r \rightarrow t \rightarrow s$ or $s \rightarrow t \rightarrow r$. Without loss of generality we can assume that $r \rightarrow t \rightarrow s$. Then Lemma 5.8 (3) ensures that $r \rightarrow u \rightarrow s$ or $s \rightarrow u \rightarrow r$ for every $u \in T$, whence, using Lemma 5.8 (4), it follows that $\neg \exists u \in T (r \rightarrow u \leftarrow s \vee r \leftarrow u \rightarrow s)$.

(4) We have shown in Lemma 5.8 (1) that there exist $r, s \in S$ and a $t \in T$ such that $r \rightarrow t \rightarrow s$. Then $r/\gamma_T(S) \neq s/\gamma_T(S)$ and thus $\gamma_T(S)$ has at least two blocks. Take any $q \in S$. From Lemma 5.8 (4) it follows that $q \sim t$. From (3) we now easily infer that $q \rightarrow t$ implies $q/\gamma_T(S) = r/\gamma_T(S)$, while $t \rightarrow q$ implies $q/\gamma_T(S) = s/\gamma_T(S)$. Therefore, $\gamma_T(S)$ has precisely two blocks.

(5) Take any $s \in S_1$ and any $t \in T_1$. Then $s \sim t$ according to Lemma 5.8 (4). Without loss of generality we can assume that $s \rightarrow t$. Let us show that $S_1 \rightrightarrows T_1$. Take any $r \in S$ and any $u \in T$. From (2) we conclude that $r \rightarrow t$ since $(r, s) \in \gamma_T(S)$. From $r \rightarrow t$ and $(t, u) \in \gamma_S(T)$ we infer $r \rightarrow u$. Therefore, $S_1 \rightrightarrows T_1$. Now, T_1 and T_2 are distinct classes of $\gamma_S(T)$, so $S_1 \rightrightarrows T_1$ implies $T_2 \rightarrow S_1$, and using the same argument as above we can show that $T_2 \rightrightarrows S_1$. Analogously, $T_1 \rightrightarrows S_2$ and $S_2 \rightrightarrows T_2$. \square

Lemma 5.10 *Let D be a finite homomorphism-homogeneous reflexive improper bidirectionally disconnected digraph, and let S , T and U be three distinct classes of $\theta(D)$ such that $S \rightrightarrows T$ and $T \rightrightarrows U$. Then $\gamma_S(T) = \gamma_U(T)$, that is, $\gamma_S(T)$ does not depend on S .*

Proof. Assume, to the contrary, that $\gamma_S(T) \neq \gamma_U(T)$ and that there exists a pair $(t, t') \in \gamma_S(T)$ such that $(t, t') \notin \gamma_U(T)$. Then the definition of γ and Lemma 5.9 provide us with an $s \in S$ and a $u \in U$ so that $t \rightarrow s \leftarrow t'$ and $t \rightarrow u \rightarrow t'$. The mapping

$$f : \begin{pmatrix} s & t & u \\ s & t & s \end{pmatrix}$$

is a homomorphism between finitely induced substructures of D , so it extends to an endomorphism f^* of D . From $t' \rightrightarrows t$ it follows that $f^*(t') \rightrightarrows t$, so

$f^*(t') \in T$. On the other hand, $u \rightarrow t' \rightarrow s$ implies $f^*(t') \rightleftharpoons s$, so $f^*(t') \in S$. Contradiction. \square

Theorem 5.11 *Let D be a finite reflexive bidirectionally disconnected improper digraph with back-and-forth. Then D is homomorphism-homogeneous if and only if D is an inflation of a homomorphism-homogeneous digraph with involution.*

Proof. (\Rightarrow) Assume that D is homomorphism-homogeneous. For a class S of $\theta(D)$, define $\gamma(S)$ as follows:

- if $S \rightleftharpoons T$ for some class T of $\theta(D)$ distinct from S , let $\gamma(S) = \gamma_T(S)$;
- if $S \rightleftharpoons T$ for no class T of $\theta(D)$ distinct from S , let $\gamma(S) = S^2$.

With $\gamma(S)$ defined for every $S \in V(D)/\theta(D)$, we define $\gamma(D)$ by

$$\gamma(D) = \bigcup_{S \in V(D)/\theta(D)} \gamma(S).$$

Then $D/\gamma(D)$ is well defined (Lemmas 5.10 and 5.9) and it is a retract of D (Lemma 5.1), so $D/\gamma(D)$ is homomorphism-homogeneous. Moreover, from Lemma 5.9 (4) and (5), $D/\gamma(D)$ is a digraph with involution since every $\theta(D)$ -class consists of at most two $\gamma(D)$ -classes.

(\Leftarrow) Let D be an inflation of a homomorphism-homogeneous digraph with involution. According to Lemma 2.1 it suffices to show that $D[S] \Rightarrow D[S']$ for all connected components S, S' of D . Let S and S' be connected components of D and let $f : U \rightarrow W$ be a homomorphism from $D[U]$ to $D[W]$ where $U \subseteq S$ and $W \subseteq S'$. Let $D = R\langle V_1, \dots, V_n \rangle$ be an inflation of R where R is one of the digraphs listed in the statement of Theorem 4.10, and let $\gamma(D)$ be the equivalence relation on $V(D)$ whose blocks are V_1, \dots, V_n , so that $D/\gamma(D) \cong R$. Therefore, R is a retract of D , so there exists a retraction-projection pair $r : V(D) \rightarrow V(R)$ and $j : V(R) \rightarrow V(D)$ such that $r \circ j = \text{id}$.

Assume, first, that for every $\gamma(D)$ -class F there exists a $\gamma(D)$ -class F' such that $f(U \cap F) \subseteq F'$. Then $g : U/\gamma(D) \rightarrow W/\gamma(D)$ defined by $g(u/\gamma(D)) = f(u)/\gamma(D)$ is well-defined and it is a homomorphism from $R[U/\gamma(D)]$ to $R[W/\gamma(D)]$. We know that R is homomorphism-homogeneous,

so there exists a homomorphism g^* from $R[S/\gamma(D)]$ to $R[S'/\gamma(D)]$ which extends g . But then the mapping $f^* : S \rightarrow S'$ defined by

$$f^*(x) = \begin{cases} f(x), & x \in U \\ j \circ g^* \circ r(x), & x \in S \setminus U \end{cases}$$

is a homomorphism from $D[S]$ to $D[S']$ which extends f .

Assume, now, that there exists a $\gamma(D)$ -class F such that $f(U \cap F)$ spreads over at least two $\gamma(D)$ -classes. Then $f(U \cap F)$ spreads over exactly two $\gamma(D)$ -classes F_1, F_2 which belong to the same $\theta(D)$ -class H , that is, $H = F_1 \cup F_2$. Choose $x_1, x_2 \in U \cap F$ in such a way that $f(x_1) \in F_1$ and $f(x_2) \in F_2$. Let us show that $f(U) \subseteq H$. Clearly, if T is a $\theta(D)$ -class that contains F then $f(T \cap U) \subseteq H$. Let, now, T be a $\theta(D)$ -class such that $T \cap U \neq \emptyset$ and $F \cap T = \emptyset$. Take any $y \in T \cap U$. Since D is an inflation of R it follows that $y \rightrightarrows \{x_1, x_2\}$ or $\{x_1, x_2\} \rightrightarrows y$, say $y \rightrightarrows \{x_1, x_2\}$. Then $f(y) \rightrightarrows \{f(x_1), f(x_2)\}$. Since $f(x_1)$ and $f(x_2)$ belong to distinct $\gamma(D)$ -classes of the same $\theta(D)$ -class, if $f(y) \notin H$ then $f(x_1) \rightarrow y$ and $f(x_2) \not\rightarrow y$, or $f(x_2) \rightarrow y$ and $f(x_1) \not\rightarrow y$. Therefore, $f(y) \in H$. This shows that $f(U) \subseteq H$. Since $D[H] \cong K_{n_i}^\circ$, it is now easy to extend f to a homomorphism f^* from $D[S]$ to $D[S']$: take any $h \in H$ and define f^* by

$$f^*(x) = \begin{cases} f(x), & x \in U \\ h, & x \in S \setminus U. \end{cases}$$

This concludes the proof. \square

Corollary 5.12 *Let D be a finite reflexive binary relational system. If D is bidirectionally disconnected, then D is homomorphism-homogeneous if and only if*

- (1) D is a finite homomorphism-homogeneous quasiorder; or
- (2) D is an inflation of $k \cdot C_3^\circ + l \cdot \mathbf{1}^\circ$ for some $k, l \geq 0$ such that $k + l \geq 1$;
or
- (3) D is an inflation of a homomorphism-homogeneous digraph with involution.

If D is bidirectionally connected then the problem of deciding whether D is homomorphism-homogeneous is coNP-complete.

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